# The Intrinsic Hodge Theory of p-adic Hyperbolic Curves

by Shinichi Mochizuki

**Contents:**



## §1. Uniformization Theory as a Hodge Theory at Arithmetic Primes

### (A.) Uniformization as a Catalogue of Rational Points

We begin our discussion by posing the following elementary problem concerning algebraic varieties over the complex numbers (where, roughly speaking, an "algebraic variety over the complex numbers" is a geometric object defined by polynomial equations with coefficients which are complex numbers):

**Problem:** Given an *algebraic variety* $Z$ over $\mathbf{C}$, it is possible to give some sort of *natural explicit catalogue* of the rational points $Z(\mathbf{C})$ of $Z$?

To gain a sense of what is meant by the expression "a natural explicit catalogue," it is useful to begin by thinking about some basic examples. Perhaps the simplest nontrivial examples of algebraic varieties are *plane curves*, i.e., subvarieties of $\mathbf{A}^2_{\mathbf{C}}$ (two-dimensional affine space over $\mathbf{C}$) defined by a single polynomial equation

$$f(X, Y) = 0$$

in two variables. In this case, the set of rational points $Z(\mathbf{C})$ of the corresponding variety $Z$ is given by

$$Z(\mathbf{C}) = \{(x, y) \in \mathbf{C}^2 | \;\; f(x, y) = 0\}$$

Moreover, we can classify plane curves by the degree of the defining equation $f(X, Y)$. We then see that the resulting sets $Z(\mathbf{C})$ may be explicitly described as follows:

(1.) *The Linear Case* ($\deg(f) = 1$): Up to coordinate transformations, this is the case given by the equation $f(X, Y) = X$. In this case, we then obtain an explicit catalogue of the rational points by:

$$(0, ?) : \mathbf{C} \;\; \stackrel{\sim}{\to} \;\; Z(\mathbf{C})$$

(i.e., mapping $z \in \mathbf{C}$ to $(0, z) \in Z(\mathbf{C})$).

(2.) *The Quadratic Case* ($\deg(f) = 2$): Up to coordinate transformations (and ruling out degenerate cases), we see that this is essentially the case where the equation $f(X, Y) = X \cdot Y - 1$. In this case, an explicit catalogue is given by the *exponential map*:

$$\exp : \mathbf{C} \;\; \to \;\; Z(\mathbf{C}) = \mathbf{C}^{\times}$$

(In fact, the map may be defined intrinsically, without using coordinate transformations to render the defining equation in the "standard form" $X \cdot Y = 1$.)

(3.) *The Cubic Case* ($\deg(f) = 3$): Up to adding the point(s) at infinity, this is essentially the case where we are dealing with an *elliptic curve* $E$. In this case, as well, we have a natural exponential map:

$$\exp_E : T_E \;\; \to \;\; E(\mathbf{C})$$

(where $T_E$ is the one-dimensional complex vector space given by the tangent space to some fixed point – "the origin" – of $E$). This map allows us to think of $E$ as being of the form "$\mathbf{C}/\Lambda$" (where $\Lambda \cong \mathbf{Z}^2$ is a lattice in $\mathbf{C}$).

(4.) <u>*Higher Degree*</u>: If $\deg(f) \geq 4$, or we wish to consider lower degree cases with lots of points removed, then we are led naturally to the following notion:

A *hyperbolic curve* $Z$ is a smooth, proper connected algebraic curve of genus $g$ with $r$ points removed, where we assume that $2g - 2 + r > 0$.

According to the *uniformization theorem of Köbe*, hyperbolic curves may be uniformized by the upper half-plane $\mathfrak{H} \stackrel{\text{def}}{=} \{z \in \mathbf{C} \mid \mathrm{Im}(z) > 0\}$, i.e., we have a surjective (holomorphic) covering map:

$$\mathfrak{H} \to Z(\mathbf{C})$$

which allows us to think of $Z(\mathbf{C})$ as being of the form $\mathfrak{H}/\Gamma$, where $\Gamma$ is some discrete group acting on $\mathfrak{H}$.

KOEBE.EPS

**(B.) "Intrinsic" Hodge Theories**

In the preceding section, we posed the problem of *explicitly cataloguing the rational points of a variety* (over $\mathbf{C}$). By looking at various examples, we saw that this problem may also be worded – in perhaps more familiar terms – as the problem of finding natural *uniformizations* of varieties. In the present section, we would like to further refine our understanding of the problem of finding natural uniformizations/explicit catalogues of rational points by rewording this problem in terms of the language of "Hodge theory."

First, let us discuss what we mean in general by the notion of a "*Hodge theory.*" By a Hodge theory, we shall mean an *equivalence* of the following form:

$$\begin{pmatrix} \text{algebraic} \\ \text{geometry} \\ \text{(e.g.,} \\ \text{rational} \\ \text{points)} \end{pmatrix} \iff \begin{pmatrix} \underline{\text{Over } \mathbf{C}}\text{: topology +} \\ \text{differential geometry} \\ \\ \underline{\text{Over the } p\text{-adics}}\text{:} \\ \text{pro-}p \text{ étale} \\ \text{topology + Galois action} \end{pmatrix}$$

The most familiar example of such an equivalence is the *Hodge theory of cohomologies*. Over $\mathbf{C}$, this amounts to "classical Hodge theory," i.e., the well-known isomorphism between the de Rham cohomology of an algebraic variety (which is well-known to be an *algebro-geometric* invariant of the variety) and the singular cohomology of the variety (which is a *topological* invariant). More recently, the *p-adic Hodge theory of cohomologies* has been developed by Fontaine et al. (cf., e.g., [Falt2,3]). This theory asserts an equivalence between the (algebraic) de Rham cohomology of an algebraic variety over a finite extension $K$ of $\mathbf{Q}_p$ and the $p$-adic étale cohomology of the variety, equipped with its natural Galois action (i.e., action of $\mathrm{Gal}(K)$).

This "Hodge theory of cohomologies" is the most basic example of a "Hodge theory" as defined above. In the present manuscript, however, we would like to consider a different kind of Hodge theory which we shall call an *intrinsic Hodge theory*, or IHT, for short. By an intrinsic Hodge theory, we mean a Hodge theory – i.e., an equivalence of the form discussed above – in which the invariant which appears on the "algebraic geometry" side is the "*variety itself*."

There are several different ways to interpret the phrase "the variety itself." In the present manuscript, we shall consider the following two interpretations:

(1.) <u>The Physical Interpretation</u>: In this interpretation, one takes the phrase the "variety itself" to mean the "rational points of the variety."

(2.) <u>The Modular Interpretation</u>: In this interpretation, one takes the phrase the "variety itself" to mean the "moduli of the variety."

Note that (it is a tautology of terminology that) a physical IHT essentially amounts to some sort of explicit description of the rational points of the variety in terms of topology and geometry. Thus, one may summarize the above discussion as follows:

physical IHT = uniformization of the variety

modular IHT = uniformization of the moduli space of deformations of the variety

Before concluding this section, we make some remarks on the relationship between the "IHT's" that we wish to discuss here and the more well-known "Hodge theories of cohomologies." First of all, although in general, IHT's are not the same as Hodge theories of cohomologies, typically in proving theorems which realize IHT's, the technical tools of the corresponding Hodge theory of cohomologies (e.g., in the $p$-adic case, the techniques of so-called "$p$-adic Hodge theory" as in [Falt2]) play an important role. Secondly, in the case of $\mathbf{G}_{\mathrm{m}} = V(X \cdot Y - 1)$ (i.e., Example (2) in §1, (A.)), as well as in the case of elliptic curves (i.e., Example (3) in §1, (A.)), it just so happens that the first cohomology module of the curve "is" the curve itself, i.e., in more sophisticated language, in these cases the curve in question is a 1-*motive*. Thus, in these cases, it turns out that the notions of IHT and Hodge theory of $H^1$ *coincide*. In particular, in these cases, the well-known Hodge theory of $H^1$ already provides a uniformization of the curve. Note that this differs quite substantially from the case of higher genus (Example (4) in §1, (A.)).

### (C.) Completion at Arithmetic Primes

So far in our discussion, we have ignored the important issue of what sort of *ground field* should be considered in our discussion of uniformizations/explicit catalogues of rational points/IHT's. In the examples of §1, (A.), we considered the situation over *the complex number field*, since this is the most elementary and well-known example of a ground field over which IHT's may be realized.

Of course, ideally, one would like to realize IHT's over any field, for instance, over a *number field* (i.e., a finite extension of $\mathbf{Q}$) – a case for which the problem of determining the set of rational points explicitly is of prime interest. Unfortunately, however, typically, in order to realize an IHT (or, indeed, any sort of "Hodge theory"), one must work over a base which is complete with respect to some sort of "arithmetic prime." The three main examples of this sort of base are the following:

(i.) <u>the complex number field $\mathbf{C}$</u> (this also covers the case of $\mathbf{R}$ by working with objects over $\mathbf{C}$ equipped with an action of complex conjugation, i.e., of $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$)

(ii.) <u>a $p$-adic field $K$</u> (i.e., a finite extension of $\mathbf{Q}_p$)

(iii.) <u>power series over $\mathbf{Z}$</u> – typically arising as the completion of some sort of moduli space at a $\mathbf{Z}$-valued point corresponding to a *degenerate object* parametrized by the moduli space.

Indeed, all completions of number fields fall under cases (i.) and (ii.). Thus, if one is ultimately interested in rational points of number fields, IHT's over bases as in (i.) and (ii.) are of natural interest. Also, since the coefficients of the powers series appearing in (iii.) are integers, case (iii.) is also of substantial arithmetic interest.

In the following discussion, the following principle will serve as an important guide:

<u>Guiding Principle</u>: For every type of arithmetic prime (i.e., cases (i.), (ii.), and (iii.) above), one expects that there should be a *canonical uniformization theory* at that prime.

In general, however, one does not expect that the canonical uniformization theories at different primes should be compatible with one another. We will return to this point in more detail later.

In terms of types of varieties, the main cases in which one has well-developed physical and modular IHT's are the following:

(1.) abelian varieties

(2.) hyperbolic curves.

The physical and modular IHT's in these cases may be roughly summarized in the following charts:

(1.) Abelian Varieties

| **C** | *p*-adic | Degenerate Object |
|---|---|---|
| exponential map of abelian varieties/ Siegel upper half-plane uniformization | Tate's theorem/ Serre-Tate theory | Schottky uniformizations of Tate/ Mumford |

(2.) Hyperbolic Curves

| **C** | *p*-adic | Degenerate Object |
|---|---|---|
| Fuchsian uniformization/ uniformizations of Teichmüller and Bers | §2 (anabelian conjecture)/ §3 ($p$-adic Teichmüller theory) | formal algebraic Schottky uniformization of Mumford |

Of these two examples, undoubtedly the example of abelian varieties is the more well-known. *Over* **C**, the exponential map of an abelian variety gives a uniformization of the abelian variety by a complex linear space. This generalizes Example (3) of §1, (A.). Moreover, by using the periods that one obtains from this uniformization, one obtains a uniformization of the moduli space of abelian varieties by the Siegel upper half-plane. Thus, we see both the *physical* and *modular* aspects of the IHT of abelian varieties in evidence.

In the *p-adic case*, the IHT of an abelian variety essentially amounts to the $p$-adic Hodge theory of $H^1$ of the abelian variety. Although the $p$-adic Hodge theory of $H^1$ of an abelian variety has many different aspects, most of these may be traced to the fundamental paper of Tate ([Tate]) in the late 1960's. In this paper, the main theorem ("Tate's theorem" in the chart) states that homomorphisms between formal groups (e.g., the formal groups arising from abelian varieties) over $p$-adic fields are essentially equivalent to homomorphisms between the corresponding Tate modules. In some sense, this result is the analogue for abelian varieties of the main theorem (Theorem 1) of §2 below, and may be regarded as a sort of physical IHT for abelian varieties. On the other hand, the modular aspect of the IHT of abelian varieties may be seen most easily in Serre-Tate theory, which gives rise to $p$-adic parameters on the moduli space of abelian varieties that are very much analogous to the Siegel upper half-plane uniformization in the complex case.

Finally, in a neighborhood of a point in the moduli space corresponding to a *degenerating abelian variety*, one has the theory of Tate curves, generalized by Mumford and Faltings/Chai ([FC]). Moreover, it turns out that in the case of abelian varieties, the complex, $p$-adic, and degenerating object theories are all compatible with one another. For instance, if one specializes the uniformizing parameters that one obtains on the moduli space in a neighborhood of a point corresponding to a degenerating abelian variety to a $p$-adic (respectively, complex) base, one obtains parameters compatible with the Serre-Tate parameters (respectively, the Siegel upper half-plane uniformization).

Next, we consider the case of *hyperbolic curves. Over* **C***,* the physical aspect of the IHT of hyperbolic curves (cf. Example (4) of §1, (A.)) essentially amounts to the Fuchsian uniformization. Then just as the exponential map uniformization of an abelian variety "induces" the Siegel upper half-plane uniformization of the moduli space of abelian varieties, the Fuchsian uniformization of a hyperbolic curve "induces" the Bers uniformization of the moduli space of hyperbolic curves (cf. §3, (A.) below, as well as the Introduction of [Mzk4]).

In a neighborhood of a point in the moduli space corresponding to a totally degenerate (proper) hyperbolic curve, one has the theory of [Mumf]. Note, however, that this theory is *not* compatible with the theory of the Fuchsian uniformization in the following sense: If one specializes the (**Z**-coefficient) power series in the base ring to some **C**-valued point in a neighborhood of a point in the moduli space corresponding to a totally degenerate curve, the resulting uniformization over **C** that one obtains is the so-called *Schottky uniformization* of the curve. This uniformization is completely different from the Fuchsian uniformization.

Finally, we come to the *$p$-adic case.* It seems that the IHT of $p$-adic hyperbolic curves has not been studied extensively until relatively recently ([Mzk1-5]). Thus,

> *It is the goal of this manuscript to report on recent developments concerning the intrinsic Hodge theory (IHT) of $p$-adic hyperbolic curves.*

The physical aspect, which concerns a theorem (Theorem 1) that gives a strong solution to Grothendieck's anabelian conjecture, will be the topic of §2. The modular aspect, which concerns a theory – which we call *$p$-adic Teichmüller theory* – which may be regarded as either the hyperbolic curve analogue of Serre-Tate theory or the $p$-adic analogue of the theory of the Fuchsian and Bers uniformizations, will be discussed in §3. We remark here that this $p$-adic Teichmüller theory is *not* compatible with the specialization of the theory of [Mumf] to the $p$-adic case. This may disappoint some readers, but is, in fact, natural in view of the fact that even over **C**, the specialization of the theory of [Mumf] to the complex numbers is not compatible (as remarked above) with the theory of the Fuchsian uniformization. Moreover, it is in line with the general Guiding Principle discussed above that to each sort of arithmetic prime there should correspond a natural uniformization theory of hyperbolic curves. Thus, it seems to the author that it is meaningless to argue as to whether it is the specialization of the [Mumf] to the $p$-adic case or the theory of [Mzk1-4] which is the "true" $p$-adic analogue of the Fuchsian uniformization. That is to say, it seems more natural to the author to regard the theory of [Mumf] as the "true" analogue of the Fuchsian uniformization at the "degenerating object prime," and the theory of [Mzk1-4] as the "true" analogue of the Fuchsian uniformization at "the prime $p$."

## §2. The Physical Aspect: Embedding by Automorphic Forms

**(A.) The Complex Case**

**(B.) The Arithmetic Fundamental Group**

**(C.) The Main Theorem**

**(D.) Comparison with the Case of Abelian Varieties**

## §3. The Modular Aspect: Canonical Frobenius Actions

**(A.) The Complex Case**

**(B.) Teichmüller Theory in Characteristic p**

**(C.) Canonical p-adic Liftings**

## Bibliography


[BK] Bloch, S. and Kato, K., $L$-Functions and Tamagawa Numbers of Motives in *The Grothendieck Festschrift*, Volume I, Birkhäuser (1990), pp. 333-400.

[Falt1] Faltings, G., Endlichkeitssätze für Abelschen Varietäten über Zahlkörpern, *Inv. Math.* **73** (1983), pp. 349-366.

[Falt2] Faltings, G., $p$-adic Hodge Theory, *Journal of the Amer. Math. Soc.* **1**, No. 1 (1988), pp. 255-299.

[Falt3] G. Faltings, Crystalline Cohomology and p-adic Galois Representations, *JAMI Conference*, Johns Hopkins Univ. Press (1990), pp. 25-79.

[FC] G. Faltings and C.-L. Chai, *Degenerations of Abelian Varieties*, Springer Verlag (1990).

[Ihara] Ihara, Y., On the Differentials Associated to Congruence Relations and the Schwarzian Equations Defining Uniformizations, *Jour. Fac. Sci. Univ. Tokyo*, Sect. IA Math. **21** (1974), pp. 309-332.

[LS] Lochak, P. and Schneps, L., *Geometric Galois Actions: 1. Around Grothendieck's Esquisse d'un Programme*, London Math. Soc. Lect. Note Ser. **242**, Cambridge Univ. Press, 1997.

[Mumf] D. Mumford, An Analytic Construction of Degenerating Curves over Complete Local Rings, *Compositio Math.* **24**, pp. 129-174 (1972).

[Mzk1] Mochizuki, S., A Theory of Ordinary $p$-adic Curves, RIMS Preprint 1033 (September 1995); *Publ. of RIMS* **32**, No. 6 (1996), pp. 957-1151.

[Mzk2] Mochizuki, S., The Generalized Ordinary Moduli of $p$-adic Hyperbolic Curves, RIMS Preprint 1051 (December 1995); 281 pp.

[Mzk3] Mochizuki, S., Combinatorialization of $p$-adic Teichmüller Theory, RIMS Preprint 1076 (April 1996); 32 pp.

[Mzk4] Mochizuki, S., *Foundations of $p$-adic Teichmüller Theory*, to appear (as a book) from International Press.

[Mzk5] Mochizuki, S., The Local Pro-$p$ Anabelian Geometry of Curves, RIMS Preprint 1097 (August 1996); 84 pp.

[NTM] Nakamura, H., Tamagawa, A., and Mochizuki, S., The Grothendieck Conjecture on the Fundamental Groups of Algebraic Curves, *Sugaku* **50**, No.2 (1998), pp. 1-17 (in Japanese).

[Pop] Pop, F., On Grothendieck's conjecture of birational anabelian geometry II, Preprint (1995).

[SGA 1] A. Grothendieck et al., *Revêtements étales et groupe fondamental*, Lecture Notes in Mathematics **224**, Springer, 1971.

[Tama1] Tamagawa, A., The Grothendieck conjecture for affine curves, *Compositio Math.* **109**, No. 2 (1997), pp. 135-194.

[Tama2] Tamagawa, A., On the fundamental groups of curves over algebraically closed fields of characteristic $> 0$, RIMS Preprint 1182 (January 1998).

[Tate] J. Tate, *p-Divisible Groups*, Proceedings of a Conference on Local Fields, Driebergen, Springer Verlag (1967), pp. 158-183.